

Geometric Phases and Quantum Computations

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Abstract

Calculation aspects of holonomic quantum computer (HQC) are considered. Wilczek–Zee potential defining the set of quantum calculations for HQC is explicitly evaluated. Principal possibility of realization of the logical gates for this case is discussed.

The conception of quantum computer (QC) and quantum calculation developed in 80-th [1]–[2] were found to be fruitful both for computer science and mathematics as well as for physics [3]. Although a device being able to perform quantum calculations is far away from practical realization, there is a number of theoretical proposals of such a construct [4]–[16]. Recently holonomic realization of QC was proposed [17],[18]. It is based on the notion of non-Abelian Berry phase[19]. To perform quantum computations in this approach one needs a parametrically driven quantum system described by the Hamiltonian $H(\boldsymbol{\lambda})$ with adiabatically evolving parameters $\boldsymbol{\lambda}(t) = (\lambda_1(t), \dots, \lambda_N(t))$. Adiabaticity means that $\Omega \ll \omega_{min}$ where Ω is the characteristic frequency in the Fourier spectrum of $\boldsymbol{\lambda}(t)$ and ω_{min} is the minimal transition frequency in the spectrum of $H(\boldsymbol{\lambda})$. If the spectrum is degenerate then the cyclic evolution of N parameters $\lambda_A(t)$ results in a unitary transformation of each eigenspace of H . Such transformations can be treated as computations on the eigenspace representing a part of the qubit space of HQC. These computations are shown to form a universal set of quantum gates for a specific relevant model [20]. For instance, one should choose of independent 2-loops \mathcal{C} in the control manifold $M = \{\text{all possible } \boldsymbol{\lambda}\}$ and then the non-Abelian Berry phases $U(\mathcal{C})$ corresponding to each loop represent all basic gates. To produce such gates one should have a possibility to control i.e. for each closed adiabatic curve \mathcal{C} in M one is to know the explicit form of $U(\mathcal{C})$. To know that it is necessary to be aware of an explicit expression of non-Abelian Wilczek–Zee potential \mathbf{A} which determines $U(\mathcal{C})$ completely. A more or less universal method of explicit calculation of \mathbf{A} is to our knowledge up to now absent. For an early overview on this subject see [21]. A variant of Berry phase evaluation for a finite level system relevant to HQC was proposed in [18] where a specific parametrization of $SU(n)$ is used. In this paper we present some less general but to our mind some more effective method of evaluation of non-Abelian Berry phase which can appear to be useful for HQC. For this purpose we restrict ourselves with systems

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with finite number energy levels described by isospectral Hamiltonians, i.e. by ones whose spectrum does not depend on time via adiabatical evolution of their parameters. At first we demonstrate the method for the Abelian case, then derive the result for the non-Abelian one and in conclusion some possibilities of physical realization of the basic gates.

In view of above made restrictions the Hamiltonian we consider can be present in the form

$$H(t) = V(t)H_0V^\dagger(t), \quad (1)$$

where the $n \times n$ matrix H_0 does not depend on time. As the qubit space must be large enough then the state space of H must be so as well. In this case derivation of \mathbf{A} becomes difficult even if there are few independently evolving parameters that is the case for a typical experimental situation.

We derive an explicit closed expression for \mathbf{A} in terms of $H_{ij}(t)$ that have a direct physical sense. Let us consider the Hamiltonian $H(\boldsymbol{\lambda})$ with $\lambda_A(t)$ evolving adiabatically thus Born–Fock theorem [22] is valid:

$$H(\boldsymbol{\lambda})\varphi_i^{a_i}(\boldsymbol{\lambda}) = E_i(\boldsymbol{\lambda})\varphi_i^{a_i}(\boldsymbol{\lambda}), \quad i = 1, \dots, s, \quad a_i = 1, \dots, d_i. \quad (2)$$

Here d_i is the degeneration rate and $\sum_{i=1}^s d_i = n$. It should be emphasized that the finite-matrix form of the Hamiltonian $H(\boldsymbol{\lambda})$ may be not only the result of an approximation, i.e. one neglects all energy levels whose number is greater than n but also a consequence of the symmetry of the system under consideration. This means if the Hamiltonian \hat{H} we start with possesses a Liean symmetry, i.e. it can be represented in the form $\hat{H} = c_A(\boldsymbol{\lambda})\hat{T}_A$, where \hat{T}_A are generators of a Lie algebra, then the state space of the system is split in a set of components irreducible with respect to the algebra and for each finite-dimensional component we come to the finite-matrix form.

Cyclic evolution of $\boldsymbol{\lambda}$ results [19] in a monodromy transformation of $\boldsymbol{\varphi} = (\varphi_i^1, \dots, \varphi_i^{d_i})$:

$$\boldsymbol{\varphi}(T) = U(\mathcal{C})\boldsymbol{\varphi}(0), \quad (3)$$

where t parametrizes the curve \mathcal{C} so that $\boldsymbol{\lambda}(T) = \boldsymbol{\lambda}(0)$.

For an illustration we start with Abelian Berry phase that was firstly predicted by Berry [23]. In this case all $d_i = 1$ and instead of (3) the relation

$$\varphi_m(T) = e^{i\gamma_m(\mathcal{C})}\varphi_m(0), \quad (4)$$

where

$$\gamma_m(\mathcal{C}) = \int_{\mathcal{C}} A_m, \quad A_m = i \langle \varphi_m | d\varphi_m \rangle. \quad (5)$$

As the spectrum of the Hamiltonian is finite, the ket $|\varphi_m\rangle$ is a unit vector \mathbf{m} in \mathbf{C}^n , so A_m is

$$A_m = \frac{i}{2} (\mathbf{m}^* d\mathbf{m} - \mathbf{m} d\mathbf{m}^*). \quad (6)$$

As the evolution is adiabatic, the spectrum of $H(t)$ remains always nondegenerate if it was so at the initial time. Then there is always a nonzero main minor of $H - E_m$ which we assume to consist always of the first $n - 1$ lines and columns of $H - E_m$. Denoting the matrix consisting of the first $n - 1$ lines and columns of H by H_\perp we come to the condition

$$\det(H_\perp - E_m) \neq 0 \quad (7)$$

Making use of this condition one can represent \mathbf{n} in the uniform coordinates

$$\mathbf{m} = \frac{(\boldsymbol{\xi}_m, 1)}{\sqrt{1 + |\boldsymbol{\xi}_m|^2}}$$

and express $\boldsymbol{\xi}_m$ in terms of H_{ij} for $1 \leq i, j \leq n - 1$ and E_m :

$$\boldsymbol{\xi}_m = (H_\perp - E_m)^{-1} \mathbf{h}, \quad h_i = -H_{in}, \quad (8)$$

where \mathbf{h} is a vector in \mathbf{C}^{n-1} but not in \mathbf{C}^n . Thus we have for A_m

$$A_m = \frac{i}{2} \frac{(\boldsymbol{\xi}_m^* d\boldsymbol{\xi}_m - \boldsymbol{\xi}_m d\boldsymbol{\xi}_m^*)}{1 + |\boldsymbol{\xi}_m|^2}, \quad (9)$$

where $\boldsymbol{\xi}_m$ is completely determined by (8). Note that the result obtained is purely geometrical because it can be expressed of the Kählerian potential

$$F = \log(1 + |\mathbf{z}|^2)$$

which determines all the geometrical properties of the state space

$$SU(n) / \underbrace{U(1) \times \dots \times U(1)}_{n-1 \text{ times}}.$$

Here \mathbf{z} is a vector in \mathbf{C}^{N^2} consisting of $n(n - 1)/2$ independent components of all $\boldsymbol{\xi}_m$. One more observation is that for the case $n = 2$ corresponding to spin in the magnetic field (9) gives well known result [23]

$$A_\pm = \pm \frac{1}{2} \oint_C \frac{\xi^* d\xi - \xi d\xi^*}{1 + |\xi|^2} = \pm \frac{1}{2} \Omega(\mathcal{C}),$$

where \pm labels spin-up and spin-down states, ξ is thought of as a stereographic projection of a point on the unit sphere S^2 and $\Omega(\mathcal{C})$ is the solid angle corresponding to the given closed contour on the sphere.

Now we proceed with a more general case of degenerate spectrum. Quantum computation for this case generated by an adiabatic loop in the control manifold is determined by (3) where $U(\mathcal{C})$ is presented by a \mathcal{P} -ordered exponent

$$U(\mathcal{C}) = \mathcal{P} \exp \left(\oint_C \mathbf{A}_m \right), \quad (\mathbf{A}_m)_{ab} = i \langle m_b | dm_a \rangle. \quad (10)$$

The set of eigenvectors ξ_{ma} , $a = 1, \dots, d_m$ must obey the equation

$$(H_{\perp}^{(d_m)} - E_m) \xi_{ma} = h \mathbf{c}_a. \quad (11)$$

Here the matrix $H_{\perp}^{(d_m)}$ is constructed from the first $n - d_m$ lines and columns of H , \mathbf{c}_a are arbitrary d_m -dimensional vectors and h is the following $(n - d_m) \times d_m$ -matrix:

$$h = - \begin{pmatrix} H_{1,n-d_m+1} & \dots & H_{1,n} \\ \vdots & \dots & \dots \\ H_{n-d_m,n-d_m+1} & \dots & H_{n-d_m,n} \end{pmatrix}$$

Of course it has sense only if the condition

$$\det(H_{\perp}^{(d_m)}(t) - E_m) \neq 0 \quad (12)$$

is valid along the evolution process. The set of vectors ξ_{ma} must be orthogonalized by the standard Gram algorithm and after that we get the orthonormal set of the eigenvectors \mathbf{z}_a (here and below we have omitted the index m) in the form

$$\mathbf{z}_a = \frac{1}{\det \Gamma_{a-1}} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_a \end{pmatrix}, \quad (13)$$

where $\mathbf{x}_b = (\xi_b, \mathbf{c}_b)$ and \mathbf{c}_a is chosen to be the standard orthogonal set $\mathbf{c}_a = (0 \dots \overbrace{1}^a \dots 0)$. The matrices Γ_a are determined by

$$\Gamma_a = \begin{pmatrix} 1 + \langle \xi_1 | \xi_1 \rangle & \dots & \langle \xi_1 | \xi_a \rangle \\ \vdots & \ddots & \vdots \\ \langle \xi_a | \xi_1 \rangle & \dots & 1 + \langle \xi_a | \xi_a \rangle \end{pmatrix} = 1 + Z_a^\dagger Z_a, \quad (14)$$

where the $(n - d_m) \times a$ -matrix Z_a consists of a first lines of the $(n - d_m) \times d_m$ -matrix $Z = (H_{\perp}^{(d_m)} - E_m)^{-1} h$. Using (14) and (13) we come to the final expression for the matrix-valued 1-form \mathbf{A} :

$$\mathbf{A} = \frac{i}{2} \frac{g_{ab}^{ij} (\xi_j^* d\xi_i - d\xi_j^* \xi_i) + 2\omega_{ab}}{\det(1 + Z_{a-1}^\dagger Z_{a-1}) \det(1 + Z_{b-1}^\dagger Z_{b-1})}, \quad 1 \leq i \leq a, \quad 1 \leq j \leq b, \quad (15)$$

where

$$g_{ab}^{ij} = \Gamma_a^i \Gamma_b^{*j}, \quad \omega_{ab} = \langle \xi_j | d \operatorname{Im}(g_{ab}^{ij}) | \xi_i \rangle + \sum_{i=1}^{\min(a,b)} d \operatorname{Im}(g_{ab}^{ii}),$$

and Γ_a^i is the cofactor of ξ_i in Γ_a . Note that the change of our basis \mathbf{c}_a by $\mathbf{c}'_a = U_{ab}(\lambda) \mathbf{c}_b$ leads to a standard gauge transformation of \mathbf{A}

$$\mathbf{A}' = U \mathbf{A} U^\dagger + i(dU)U^\dagger$$

The formula (15) is the desired expression of \mathbf{A} in terms of the matrix elements of the Hamiltonian. It is correct if condition (12) is valid. It is not nevertheless a principal restriction because d_m does not depend on time due to adiabaticity of the evolution and there is always at least one nonzero $n - d_m$ -order minor of H . Then, if the minor we choose vanishes somewhere on the loop \mathcal{C} one can always take local coordinates such that the techniques considered is applicable on each segment of \mathcal{C} .

One more thing to discuss here are quantum gates and their possible physical realization. As the Hamiltonian is an element of a Lie algebra which we assume to be semisimple, generic computations are of course elements of the corresponding Lie group. We represent them in the form

$$U = \exp \left(\sum_{i=1}^k y_i H_i \right) \prod_{\alpha} \exp (\zeta_{\alpha} E_{\alpha} - \zeta_{\alpha}^* E_{-\alpha}), \quad (16)$$

where y_i are real and ζ_{α} are complex parameters, k is the dimension of the Cartan subalgebra, i labels all its linearly independent elements, the product is taking over all positive roots of the algebra, and the standard notations for Cartan–Weyl basis have been used. As the action of the first exponent in (16) results in a trivial phase factor, a generic computation is determined by the product over all roots. Thus we come to the conclusion that the elementary factors

$$U_{\alpha} = \exp (\zeta_{\alpha} E_{\alpha} - \zeta_{\alpha}^* E_{-\alpha}), \quad (17)$$

provide the basis of quantum gates for the model. Their number is e.g. for the first fundamental series A_n $n(n-1)/2$ as it should be. An idea of physical realization of the gates appears if one takes into account that for A_n E_{α} can be realized by means of ordinary bosonic creation and annihilation operators, namely $E_{\alpha} = a_i \dagger a_j$ for some $1 \leq i, j \leq n$. Then E_{α} represents nothing but two-mode squeezing operator. Thus the model considered can be applied to optical HQC with n laser beams (the case $n=2$ is considered in [20]) and the logical gates U_{α} are just two-qubit transformations realized by transformation of two laser beams.

The method presented here enables one to build in principal any computation for HQC described by a Hamiltonian with a stationary spectrum in terms of experimentally measured values, namely the matrix elements of the Hamiltonian. The method depends weakly on the dimension of the qubit space which other models based on various parametrizations of evolution operators of the system are very sensitive to. Application of this method to a concrete physical model will be discussed elsewhere.

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